Book Review

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M. BEALS (1989). *Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems*, Birkhäuser, Boston-Basel-Berlin.

The first impression of the book is that the italics type, in which the many formulas in the book are set, is rather dizzying. The reader is advised to ignore this as quickly as possible and concentrate on the mathematical contents, which are very exciting.

The book is a description of some of the phenomena which may occur in the propagation of singularities of solutions of nonlinear partial differential equations. I will try to give an impression of these phenomena, starting with the linear case with which I am more familiar.

For linear partial differential equations, with smooth coefficients, real principal symbol and simple characteristics, a theorem of Hörmander says that the singularities of the solutions propagate along the bicharacteristic curves. For the linear wave equation the latter are the light rays of classical optics, or the geodesics for the corresponding Minkowski structure; this is one of the relations between wave mechanics and Hamiltonian mechanics.

For nonlinear equations the singularities on the bicharacteristic curves in general will interact and spread out. This is related to the fact that the Fourier transform $\mathcal{F}(u \cdot v)$ of the product $u \cdot v$ of two functions u, v is equal to the convolution $\mathcal{F}u * \mathcal{F}v$ of the Fourier transforms $\mathcal{F}u$ and $\mathcal{F}v$ of u and v, respectively. As a result, if Γ and Δ are the cones of directions in which $\mathcal{F}u$ and $\mathcal{F}v$ do not decrease rapidly, then $\mathcal{F}u * \mathcal{F}v$ will not decrease rapidly in the direction of $\Gamma + \Delta$. Because the microlocal description of the singularities of a distribution w consists of looking at the directions in which the Fourier transform $\mathcal{F}(\phi w)$ does not decrease rapidly, this gives a microlocal description of the singularities of $u \cdot v$ in terms of those of u and v. In the above, ϕ is a smooth cutoff function in order to localize the variables in terms of which w is defined. When linearizing a nonlinear equation at a solution u, which has singularities, one obtains a linear equation for the variation v. However, the coefficients depend on u and will inherit its singularities. In this way the description of the singularities of products of distributions is a basic ingredient for the nonlinear theory.

One aspect of the theory is that one needs to assume a sufficient degree s of differentiability to start with (which for some people makes the theory less interesting). The new 'nonlinear' singularities caused by interaction and spreading then are of a weaker nature, with differentiability degree roughly equal to $2s$ and $3s$, respectively.

The book is a review of recent developments in the subject of the title, discussing the work of the author and many others in the field. Major stimulations were given by the work of Rauch and Reed on nonlinear hyperbolic equations in one space variable and the development of the tool of paradifferential operators by Bony. In order to give an idea of the contents, the chapters are:
I. Nonlinear microlocal analysis, II. Appearance of nonlinear singularities, III. Conormal singularities, IV. Conormal regularity after nonlinear interaction, V. Regularity and singularities in problems on domains with boundary, VI. Conormal waves on domains with boundary.

The book is written in a pleasant mixture of descriptive and technical passages. The non-expert will meet a number of concepts which are not explained in the book; some idea of the meaning may be obtained from the statements in which the concepts appear. Even when no detailed reference is given, the background material can be found in the literature which is mentioned in the extensive bibliography.